Recursion Relations for one-loop gravity amplitudes

Andreas Brandhuber, Simon McNamara, Bill Spence and Gabriele Travaglini

Centre for Research in String Theory Department of Physics
Queen Mary, University of London
Mile End Road, London, E1 4NS United Kingdom
E-mail: a.brandhuber@qmul.ac.uk, s.mcnamara@qmul.ac.uk,
w.j.spence@qmul.ac.uk, g.travaglini@qmul.ac.uk

ABSTRACT: We study the application of recursion relations to the calculation of finite one-loop gravity amplitudes. It is shown explicitly that the known five, and six graviton one-loop amplitudes for which the external legs have identical outgoing helicities, and the four graviton amplitude with helicities $-+++$ can be derived from simple recursion relations. The latter amplitude is derived by introducing a one-loop three-point vertex of gravitons of positive helicity, which is the counterpart in gravity of the one-loop three-plus vertex in Yang-Mills. We show that new issues arise for the five point amplitude with helicities $-++++$, where the application of known methods does not appear to work, and we discuss possible resolutions.

KEYWORDS: Classical Theories of Gravity, Gauge Symmetry.

Contents

1. Introduction

The twistor string proposal of Witten [1] has inspired many new techniques for the calculation of scattering amplitudes in gauge theory and gravity (see the reviews [2, 3], and references therein). Amongst these new developments, an important conceptual and practical advance was the derivation of recursion relations in gauge theories. This was achieved in [4, 5], incorporating insights from [6–9]. These BCFW recursion relations proved to be a very efficient technique for calculating scattering amplitudes, and new results at tree level in gauge theory were rapidly found [10–12]. The application to loop level amplitudes proved to be more involved however. A notable step was taken in [13], where it was shown that the introduction of a new one-loop three vertex made possible a derivation of one-loop amplitudes using recursive techniques. However, for "non-standard" cases it was shown that correction terms to naïve recursive rules were needed for the process to work; these cases occur when the amplitudes develop single poles masked by double poles as momenta are continued to complex values. It was not immediately clear how this approach could be systematised in full generality, but further work at one-loop level clarified a number of issues and made further progress [14–18].

For the case of gravity amplitudes, tree-level recursion relations have also been found [19, 20]. This involved the discovery of some new tree amplitudes and new forms of known tree amplitudes. Again it was clear that these relations were of considerable practical use, leading to much simpler derivations, as well as final forms, of amplitudes. It is natural to ask if quantum gravity amplitudes can also be studied using recursion relations. This might be relevant given recent interesting results and conjectures concerning $\mathcal{N} = 8$ supergravity (see [21–24] and references therein). In this letter we study this question, showing to what extent the new quantum gauge theory recursion techniques can be applied to gravity, and indicating where this approach breaks down and why.

As usual we write amplitudes in the spinor-helicity formalism and analytically continue the spinors to complex momenta where needed in the arguments. The BCFW recursion relations are based on two very general properties of amplitudes — analyticity and factorisation on multi-particle poles. One considers the following deformation of an amplitude which shifts the spinors of two of the n massless external particles, labelled i and j, and involves a complex parameter z,

$$\begin{aligned}
\lambda_i &\to \lambda_i \,, \\
\widetilde{\lambda}_i &\to \widetilde{\lambda}_i - z\widetilde{\lambda}_j \,, \\
\lambda_j &\to \lambda_j + z\lambda_i \,, \\
\widetilde{\lambda}_j &\to \widetilde{\lambda}_j \,.
\end{aligned} \tag{1.1}$$

This deformation does not make sense for real momenta in Minkowski space which satisfy $\widetilde{\lambda} = \pm\bar{\lambda}$, but is perfectly consistent for complex kinematics. Under the shifts (1.1), the corresponding momenta $p_i(z)$ and $p_j(z)$ remain on-shell for all complex z, and $p_i(z) + p_j(z) = p_i(0) + p_j(0)$. Hence the quantity $A(1, \ldots, p_i(z), \ldots, p_j(z), \ldots, n)$ is a well-defined on-shell amplitude for all z.[1]

The analytic structure of the z-dependent amplitude $A(z)$ is then used to calculate the physical amplitude $A(0)$. Specifically, the recursion relation can be derived from considering the following contour integral, where the contour C is the circle at infinity:

$$\frac{1}{2\pi i} \oint_C dz \frac{A(z)}{z} \,. \tag{1.2}$$

The integral in (1.2) vanishes if we assume that $A(z) \to 0$ as $z \to \infty$. It then follows from Cauchy's residue theorem that we can write the amplitude we wish to calculate, $A(0)$, as a sum of residues of $A(z)/z$:

$$A(0) = - \sum_{\substack{\text{poles of } A(z)/z \\ \text{excluding } z=0}} \text{Res}\left\{\frac{A(z)}{z}\right\} \,. \tag{1.3}$$

For tree-level Yang-Mills $A(z)$ has only simple poles. A pole at $z = z_P$ is associated with a shifted momentum $P(z) := P + z\eta$ becoming null. The residue at this pole is then obtained by factorising the shifted amplitude on this pole,

$$\text{Res}\left\{\frac{A(z)}{z}\right\} = \sum_h A_L^h(z_P) \frac{i}{P^2} A_R^{-h}(z_P) \,, \tag{1.4}$$

where the sum is over the possible assignments of the helicity h of the intermediate state. The left and right shifted amplitudes A_L and A_R are, of course, only defined for $z = z_P$ when $P(z)$ is null. The intermediate propagator is evaluated with unshifted kinematics.

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[1]One can consider more general deformations than (1.1) of course – for example more exotic shifts have shown [25] that the tree-level MHV rules [26] are an instance of BCFW recursion, and multiple shifts have been used to eliminate boundary terms in the generalisation of BCFW recursion to one-loop QCD amplitudes [13].

Since a momentum invariant involving both (or neither) of the shifted legs i and j does not give rise to a pole in z, the shifted legs i and j must always appear on opposite sides of the factorisation.

Now consider the generalisation of the BCFW recursion relations which was used to derive the rational parts of one-loop gluon amplitudes in QCD [13]. Whilst the structure of multi-particle factorisation of tree-level amplitudes implies that in general only simple poles result from performing shifts on a tree-level amplitude, this is not the case for the rational terms of one-loop Yang-Mills amplitudes. The reason for this is that in real momenta the one-loop splitting functions have only simple poles, however in complex momenta the one-loop splitting functions with helicities $+++$ and $---$ develop double poles. Since Yang-Mills tree-level amplitudes with more than three legs and less than two negative helicity gluons vanish, the one-loop amplitude with all legs of positive helicity is finite and has no multi-particle poles. Thus the shifted all-plus amplitude only has simple poles coming from the collinear singularities of the tree-level $-++$ splitting amplitude. Once shifts without a boundary term have been found, the all-plus amplitudes can then be constructed recursively by sewing all-plus loop amplitudes with fewer legs to three-gluon tree amplitudes [13].

Remarkably, the ideas of BCFW recursions are also applicable to more general QCD one-loop amplitudes such as the amplitude with a single negative-helicity gluon. As can be seen by performing the BCFW shifts on a known amplitude, there is an added complication in this one-loop recursion, as performing a shift results in the appearance of a double pole. As explained in [13] these double poles are associated with the appearance of three-point all-plus one-loop vertices. Cauchy's residue theorem does, of course, extend to this case — although the double-pole in $A(z)$ does not have a residue, we are integrating $A(z)/z$ which does have a residue,

$$\operatorname*{Res}_{z=a} \left\{ \frac{1}{z(z-a)^2} \right\} = -\frac{1}{a^2}.$$ (1.5)

Factorisation at a double pole will therefore be schematically of the form

$$A_L \frac{1}{(P^2)^2} A_R .$$ (1.6)

It is clear even on dimensional grounds that A_L and A_R cannot both be amplitudes, hence the factorisation on a double pole will involve a *vertex* with the dimensions of an amplitude times a momentum squared. This may seem puzzling at first sight, but it can be understood from the structure of the one-loop three-point vertex used for obtaining one-loop splitting amplitudes

$$A_3^{(1)}(1^+, 2^+, 3^+) = -i\frac{N_p}{96\pi^2} \frac{[12][23][31]}{K_{12}^2} .$$ (1.7)

This explicit formula shows that the three-point one-loop all-plus amplitude is either zero or infinite even in complex momenta, as it involves both the holomorphic and anti-holomorphic spinor variables. To compute the recursive double pole terms associated with the three-point all-plus factorisations, Bern, Dixon and Kosower (BDK) proposed in [13] the use

of the following vertex, which has the right dimensions and is only a function of the $\widetilde{\lambda}$ variables:

$$V_3^{(1)}(1^+, 2^+, 3^+) = -\frac{i}{96\pi^2}[12][23][31] \ . \tag{1.8}$$

In order to derive the single pole underneath the double pole, it was conjectured that the single pole differs from the double pole by a factor of the form

$$S(a_1, \hat{K}^+, a_2)\, K^2\, S(b_1, -\hat{K}^-, b_2)\,, \tag{1.9}$$

where K^2 is the propagator responsible for the pole in the shifted amplitude. The soft functions in (1.9) are given by

$$
\begin{aligned}
S(a, s^+, b) &= \frac{\langle ab \rangle}{\langle as \rangle \langle sb \rangle} \\
S(a, s^-, b) &= -\frac{[ab]}{[as][sb]} \ .
\end{aligned} \tag{1.10}
$$

The factor of K^2 in (1.9) cancels one of the two K^2 factors in the double pole term, leaving a single pole. The legs b_1 and b_2 are the external legs on the three-point all-plus vertex. Experimentation revealed that the legs a_1 and a_2 are to be identified with the external legs of the tree amplitude part of the recursive diagram which are colour adjacent to the propagator.

In the rest of the paper we will apply these ideas to study the rational one-loop amplitudes in pure Einstein gravity. In section 2 we consider the one-loop gravity amplitudes with all legs of positive helicity for the five- and six-point cases, showing that these can be correctly obtained with a suitable choice of shifts. In section 3 we derive the four graviton amplitude with all but one graviton of positive helicity. We note that double poles appear in this case, and show that the approach of [13] also works in the gravity case. Specifically, we make use of a three-point one-loop vertex of positive helicity gravitons, which generalises to the case of gravity the corresponding three-point amplitude in Yang-Mills theory. The single pole underneath the double pole has a structure which is very similar to that found for the Yang-Mills case. Perhaps surprisingly, we find that the Yang-Mills soft functions (rather than the gravity ones) are the objects in terms of which these single poles are expressed. Finally in section 4 we turn to the five graviton amplitude with all but one graviton of positive helicity. Using the techniques and rules applied successfully in the previous cases results in a formula for this amplitude which turns out not to obey the essential requirements of symmetry, and collinear and soft limit conditions. We discuss the possible sources of problems and potential resolutions.

2. The all-plus amplitude

An ansatz for the n point one-loop amplitude in pure Einstein gravity in which all the external gravitons have the same outgoing helicity was presented in [27]. This agrees with explicit computations via D-dimensional unitarity cuts for $n \leq 6$ [28]. This amplitude corresponds to self-dual configurations of the field strength, and is also related to the

one-loop maximally helicity-violating (MHV) amplitude in $\mathcal{N} = 8$ supergravity via the dimension-shifting relation of [28]. It is

$$M_n^{(1)}(1^+, 2^+, \ldots, n^+) = -\frac{i}{(4\pi)^2}\frac{1}{960}\sum_{\substack{1 \leq a < b \leq n \\ M,N}} h(a, M, b)h(b, N, a)\text{tr}[\not k_a \not k_M \not k_b \not k_N]^3 \,. \quad (2.1)$$

In this formula a and b are massless legs and M and N are two sets forming a distinct nontrivial partition of the remaining $n - 2$ legs. The first few half-soft functions $h(a, S, b)$ are given by

$$\begin{aligned}
h(a, \{1\}, b) &= \frac{1}{\langle a1 \rangle^2 \langle 1b \rangle^2}\,, \\
h(a, \{1, 2\}, b) &= \frac{[12]}{\langle 12 \rangle \langle a1 \rangle \langle 1b \rangle \langle a2 \rangle \langle 2b \rangle}\,, \\
h(a, \{1, 2, 3\}, b) &= \frac{[12][23]}{\langle 12 \rangle \langle 23 \rangle \langle a1 \rangle \langle 1b \rangle \langle a3 \rangle \langle 3b \rangle} + \frac{[23][31]}{\langle 23 \rangle \langle 31 \rangle \langle a2 \rangle \langle 2b \rangle \langle a1 \rangle \langle 1b \rangle} \\
&\quad + \frac{[31][12]}{\langle 31 \rangle \langle 12 \rangle \langle a3 \rangle \langle 3b \rangle \langle a2 \rangle \langle 2b \rangle}\,.
\end{aligned} \quad (2.2)$$

Inspection of the all-plus amplitude above shows that the standard BCFW shifts [5] give a boundary term. The following shifts, however, do not produce a boundary term:

$$\begin{aligned}
\hat{\lambda}_1 &= \lambda_1 + z[23]\eta\,, \\
\hat{\lambda}_2 &= \lambda_2 + z[31]\eta\,, \\
\hat{\lambda}_3 &= \lambda_3 + z[12]\eta\,.
\end{aligned} \quad (2.3)$$

In the recursion relation we will write for the five-point all-plus gravity amplitude we will make the convenient choice of $\eta = \lambda_4 + \lambda_5$ For the six-point all-plus amplitude recursion relation, we will set $\eta = \lambda_4 + \lambda_5 + \lambda_6$. Applying the shifts (2.3) to the all-plus amplitude (2.1) gives a shifted amplitude $M_n^{(1)}(z)$ with only simple poles. In the following we show that the residues at these poles can be computed from standard recursion relation diagrams.

2.1 The five-point all-plus amplitude

We will now use the on-shell one-loop recursion relation to re-derive the known five-point all-plus amplitude from the four-point all-plus amplitude. In this case, (2.1) becomes

$$M_4^{(1)}(1^+, 2^+, 3^+, 4^+) = -\frac{i}{(4\pi)^2}\frac{1}{60}\left(\frac{[12][34]}{\langle 12 \rangle \langle 34 \rangle}\right)^2 (s^2 + st + t^2)\,, \quad (2.4)$$

where $s = (p_1 + p_2)^2$ and $t = (p_2 + p_3)^2$. The amplitude in (2.4) was first computed using string-based methods in [29].

In the construction of the five-point all-plus amplitude, the shifts (2.3) give rise to nine different diagrams corresponding to the nine different angle brackets that the shifts can make singular. Our symmetric choice of $\eta = \lambda_4 + \lambda_5$ has the advantage that there are only two distinct types of diagrams to compute, the remaining ones being straightforward
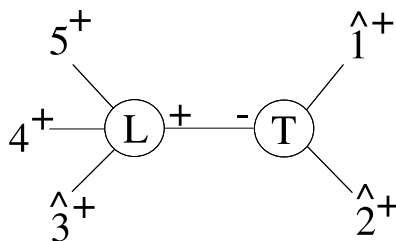
Figure 1: The diagram in the recursive expression for $M_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+)$ associated with the pole $\langle \hat{1}\hat{2} \rangle = 0$. The amplitude labelled by a T is a tree-level amplitude and the one labelled by L is a one-loop amplitude.

permutations of these two diagrams. All the recursive diagrams contain a four-point one-loop amplitude joined to a tree-level $+ + -$ amplitude.

The first type of diagram has two shifted external legs attached to the tree-level $+ + -$ diagram. There are three of these diagrams, corresponding to the simple-poles $\langle \hat{1}\hat{2} \rangle = \langle \hat{2}\hat{3} \rangle = \langle \hat{3}\hat{1} \rangle = 0$ in the shifted amplitude. The diagram associated with the pole $\langle \hat{1}\hat{2} \rangle = 0$ is drawn in figure 1.

The second type of diagram has a shifted and an unshifted leg attached to the tree-level $+ + -$ amplitude. There are six such diagrams, corresponding to the simple-poles $\langle \hat{1}4 \rangle = \langle \hat{1}5 \rangle = \langle \hat{2}4 \rangle = \langle \hat{2}5 \rangle = \langle \hat{3}4 \rangle = \langle \hat{3}5 \rangle = 0$ in the shifted amplitude. We draw in figure 2 the diagram associated with the pole $\langle \hat{1}5 \rangle = 0$.

We start by considering the diagrams of the first type. Specifically, the diagram in figure 1 contributes

$$M_4^{(1)}(\hat{3}^+, 4^+, 5^+, \hat{K}_{12}^+) \frac{i}{K_{12}} M_3^{(0)}(\hat{1}^+, \hat{2}^+, -\hat{K}_{12}^-) \,, \tag{2.5}$$

where the three-point tree amplitude is given by

$$M_3^{(0)}(1^+, 2^+, 3^-) = -i \left(i \frac{[12]^3}{[23][31]} \right)^2 \,. \tag{2.6}$$

Substituting this tree-level amplitude and the one-loop result (2.4) into (2.5) yields

$$-\frac{i}{(4\pi)^2} \frac{1}{60} \frac{[34]^4 [12]^5}{\langle 12 \rangle} \frac{\left(\langle \hat{3}4 \rangle^2 [34]^2 + \langle \hat{3}4 \rangle [34]\langle 45 \rangle [45] + \langle 45 \rangle^2 [45]^2 \right)}{\langle 5|\hat{K}|1]^2 \langle 5|\hat{K}|2]^2} \,.$$

We can eliminate $\hat{K}_{12}$ from this expression using $\langle 5|\hat{K}|1]^2 = \langle \hat{2}5 \rangle^2 [12]^2$ and $\langle 5|\hat{K}|2]^2 = \langle \hat{1}5 \rangle^2 [12]^2$. Figure 1 gives $\langle \hat{1}\hat{2} \rangle = 0$ which corresponds to a pole in the complex z-plane at $z = -\langle 12 \rangle / \langle \eta | 1 + 2 | 3]$. Then, using $\eta = \lambda_4 + \lambda_5$ gives the final contribution from figure 1 and (2.5),

$$-\frac{i}{(4\pi)^2} \frac{1}{60} \frac{[12]\left([34] - [35] \right)^4}{\langle 12 \rangle \left(\langle 14 \rangle + \langle 15 \rangle \right)^2 \left(\langle 24 \rangle + \langle 25 \rangle \right)^2} \left\{ \left(\langle 34 \rangle - \frac{\langle 12 \rangle [12]}{[34] - [35]} \right)^2 [34]^2 \right.$$

$$\left. + \left(\langle 34 \rangle - \frac{\langle 12 \rangle [12]}{[34] - [35]} \right) [34]\langle 45 \rangle [45] + \langle 45 \rangle^2 [45]^2 \right\} \,. \tag{2.7}$$
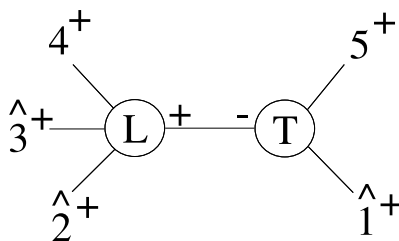
Figure 2: The diagram in the recursive expression for $M_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+)$ associated with the pole $\langle \hat{1}5 \rangle = 0$.

The contributions from the diagrams corresponding to the poles $\langle \hat{2}\hat{3} \rangle = 0$ and $\langle \hat{3}\hat{1} \rangle = 0$ are then given by cyclically permuting the external legs $\{1, 2, 3\}$.

We now consider diagrams of the second type. The diagram in figure 2 contributes

$$-\frac{i}{(4\pi)^2}\frac{1}{60}\frac{[23]^4[15]^5}{\langle 15 \rangle}\frac{\left(\langle \hat{2}\hat{3}\rangle^2[23]^2 + \langle \hat{2}\hat{3}\rangle[23]\langle \hat{3}4\rangle[34] + \langle \hat{3}4\rangle^2[34]^2\right)}{\langle 4|\hat{K}|1]^2\langle 4|\hat{K}|5]^2} \ .$$

Then, using $\eta = \lambda_4 + \lambda_5$ gives the contribution from figure 2,

$$-\frac{i}{(4\pi)^2}\frac{1}{60}\frac{[15][23]^4}{\langle 15 \rangle\langle 45 \rangle^2\left(\langle 14 \rangle + \langle 15 \rangle\right)^2}\Bigg\{\left(\langle 23 \rangle[23] + \langle 15 \rangle([14] - [15])\right)^2 \tag{2.8}$$

$$+\left(\langle 23 \rangle[23] + \langle 15 \rangle([14] - [15])\right)\left(\langle 34 \rangle[34] + \frac{\langle 15 \rangle[12][34]}{[23]}\right) + \left(\langle 34 \rangle[34] + \frac{\langle 15 \rangle[12][34]}{[23]}\right)^2\Bigg\} \ .$$

The contributions from the diagrams corresponding to the poles $\langle \hat{2}5 \rangle = 0$ and $\langle \hat{3}5 \rangle = 0$ are obtained by cyclically permuting the external legs $\{1, 2, 3\}$. The diagram corresponding to the pole $\langle \hat{1}4 \rangle = 0$ is obtained from the $\langle \hat{1}5 \rangle = 0$ diagram by interchanging legs 4 and 5. The remaining diagrams corresponding to the poles $\langle \hat{2}4 \rangle = 0$ and $\langle \hat{3}4 \rangle = 0$ are then obtained by cyclically permuting the external legs $\{1, 2, 3\}$ of the $\langle \hat{1}4 \rangle = 0$ diagram.

We have checked numerically that each of the terms in the recursion relation agree with the residues of the expression obtained by shifting the known answer (2.1) using the shifts (2.3). Hence the sum of the nine recursion relation terms is in precise agreement with the thirty terms of the known answer (2.1).

2.2 The six-point all-plus amplitude

Next we consider the six-point all-plus amplitude. We again use the shifts (2.3), but now choose $\eta = \lambda_4 + \lambda_5 + \lambda_6$. Just as in the previous five-point case, all diagrams contain a one-loop all-plus amplitude and a $+ + -$ tree-level amplitude, and there are again two types of diagrams. The first type corresponds to having two shifted legs attached to the three-point tree-level amplitude. There are three such diagrams, associated with the three poles $\langle \hat{1}\hat{2} \rangle = \langle \hat{2}\hat{3} \rangle = \langle \hat{3}\hat{1} \rangle = 0$. The diagram associated with the pole $\langle \hat{2}\hat{3} \rangle = 0$ is given in figure 3.
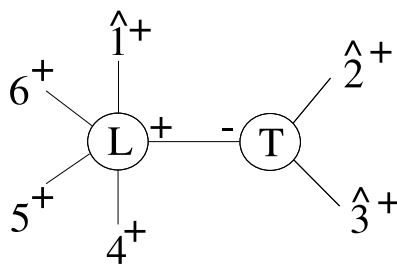
Figure 3: The diagram in the recursive expression for $M_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+, 6^+)$ associated with the pole $\langle \hat{2}\hat{3} \rangle = 0$.
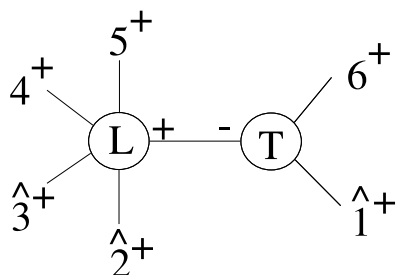

Figure 4: The diagram in the recursive expression for $M_5^{(1)}(1^+, 2^+, 3^+, 4^+, 5^+, 6^+)$ associated with the pole $\langle \hat{1}6 \rangle = 0$.

The second type of diagram corresponds to having a shifted and an unshifted leg attached to the three-point tree-level amplitude. There are nine such diagrams, associated with the nine poles $\langle \hat{1}4 \rangle = \langle \hat{1}5 \rangle = \langle \hat{1}6 \rangle = \langle \hat{2}4 \rangle = \langle \hat{2}5 \rangle = \langle \hat{2}6 \rangle = \langle \hat{3}4 \rangle = \langle \hat{3}5 \rangle = \langle \hat{3}6 \rangle = 0$. The diagram associated with the pole $\langle \hat{1}6 \rangle = 0$ is given in figure 4.

Let us start by calculating the recursive diagram in figure 3. This contributes

$$M_5^{(1)}(4^+, 5^+, 6^+, \hat{1}^+, \hat{K}_{23}^+) \frac{i}{K_{23}} M_3^{(0)}(\hat{2}^+, \hat{3}^+, -\hat{K}_{23}^-) \ . \tag{2.9}$$

The $M_5^{(1)}(+ + + + +)$ amplitude contains thirty terms (2.1). Fortunately the symmetries in the shifts and the choice for $|\eta\rangle = |4\rangle + |5\rangle + |6\rangle$ reduce these thirty terms to the following ten terms (plus the two cyclic permutations involving $\{4, 5, 6\}$ of these ten terms):

$$\frac{8i}{(4\pi)^2} \left[-\frac{[23]\langle 46 \rangle [46]^3 \langle 56 \rangle [56]^3 \big(\langle 45 \rangle [45] + \langle 56 \rangle [56] + \langle 46 \rangle [46] \big)}{\langle 23 \rangle \langle \hat{1}4 \rangle \langle \hat{1}5 \rangle \langle \hat{2}4 \rangle \langle \hat{2}5 \rangle \langle \hat{1}\hat{3} \rangle^2} \right.$$

$$-\frac{[23][14]^3[15]^3 \langle \hat{1}4 \rangle \langle \hat{1}5 \rangle \big(\langle \hat{1}4 \rangle [14] + \langle \hat{1}5 \rangle [15] + \langle 45 \rangle [45] \big)}{\langle 23 \rangle \langle 46 \rangle \langle 56 \rangle \langle \hat{2}4 \rangle \langle \hat{2}5 \rangle \langle \hat{3}6 \rangle^2}$$

$$\left. -\frac{[23][16] \big(\langle \hat{2}4 \rangle [24] + \langle \hat{3}4 \rangle [34] \big)^3 \big(\langle \hat{2}5 \rangle [25] + \langle \hat{3}5 \rangle [35] \big)^3}{\langle 23 \rangle \langle 46 \rangle \langle 56 \rangle \langle \hat{1}4 \rangle \langle \hat{1}5 \rangle \langle \hat{1}6 \rangle \langle \hat{3}4 \rangle^2 \langle \hat{2}5 \rangle^2} \right.$$

$$
\begin{aligned}
&+ \frac{[23]\langle 45\rangle[45]^3[15]^3\langle \hat{1}5\rangle\big(\langle \hat{1}4\rangle[14] + \langle \hat{1}5\rangle[15] + \langle 45\rangle[45]\big)}{\langle 23\rangle\langle 46\rangle\langle \hat{1}6\rangle\langle \hat{2}4\rangle\langle \hat{1}\hat{2}\rangle\langle \hat{3}6\rangle^2} \\
&+ \frac{[23]\langle 46\rangle[46]^3[16]^3\langle \hat{1}6\rangle\big(\langle \hat{1}4\rangle[14] + \langle \hat{1}6\rangle[16] + \langle 46\rangle[46]\big)}{\langle 23\rangle\langle 45\rangle\langle \hat{1}5\rangle\langle \hat{2}4\rangle\langle \hat{1}\hat{2}\rangle\langle \hat{3}5\rangle^2} \\
&- \frac{[23][56]\big(\langle 45\rangle[45] + \langle 56\rangle[56] + \langle 46\rangle[46]\big)^3\big(\langle \hat{2}4\rangle[24] + \langle \hat{3}4\rangle[34]\big)^3}{\langle 23\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle\langle \hat{1}5\rangle\langle \hat{1}6\rangle\langle \hat{3}\hat{1}\rangle^2\langle \hat{2}4\rangle^2} \\
&- \frac{[23][16]\langle 45\rangle[45]^3\big(\langle \hat{2}5\rangle[25] + \langle \hat{3}5\rangle[35]\big)^3}{\langle 23\rangle\langle 46\rangle\langle \hat{1}4\rangle\langle \hat{1}6\rangle\langle \hat{2}6\rangle\langle \hat{1}\hat{2}\rangle\langle \hat{3}5\rangle^2} \\
&- \frac{[23][15]\langle 46\rangle[46]^3\big(\langle \hat{2}6\rangle[26] + \langle \hat{3}6\rangle[36]\big)^3}{\langle 23\rangle\langle 45\rangle\langle \hat{1}4\rangle\langle \hat{1}5\rangle\langle \hat{2}5\rangle\langle \hat{1}\hat{2}\rangle\langle \hat{3}6\rangle^2} \\
&- \frac{[23][56][14]^3\langle \hat{1}4\rangle\big(\langle 45\rangle[45] + \langle 56\rangle[56] + \langle 46\rangle[46]\big)^3}{\langle 23\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle\langle \hat{2}5\rangle\langle \hat{2}6\rangle\langle \hat{3}\hat{1}\rangle^2} \\
&- \frac{[23][56][14]^3\langle \hat{1}4\rangle\big(\langle \hat{2}4\rangle[24] + \langle \hat{3}4\rangle[34]\big)^3}{\langle 23\rangle\langle 56\rangle\langle \hat{1}5\rangle\langle \hat{1}6\rangle\langle \hat{2}5\rangle\langle \hat{2}6\rangle\langle \hat{3}4\rangle^2} \Bigg] \, .
\end{aligned}
\tag{2.10}
$$

The diagram in figure 3 is associated with $\langle \hat{2}\hat{3}\rangle = 0$ or, equivalently, with $z = -\langle 23\rangle/\langle \eta|2 + 3|1]$. Using this value for z it is then simple to rewrite the brackets containing hatted spinors in (2.10) in terms of the unshifted spinor variables. We have checked numerically that the expression (2.10) plus the permutations agrees with the residue at $\langle \hat{2}\hat{3}\rangle = 0$ of the known amplitude (2.1). The prototypical diagram of the second type is drawn in figure 4. This contributes

$$
M_5^{(1)}(\hat{2}^+, \hat{3}^+, 4^+, 5^+, \hat{K}_{16}^+)\frac{i}{K_{16}}M_3^{(0)}(6^+, \hat{1}^+, -\hat{K}_{16}^-) \, .
\tag{2.11}
$$

Just like the other term, the $M_5^{(1)}(+ + + +)$ amplitude contains thirty terms, but the symmetries of the shifts and the choice of $|\eta\rangle = |4\rangle + |5\rangle + |6\rangle$ reduce these to the following set of terms (plus the relevant permutations):

$$
\begin{aligned}
\frac{8i}{(4\pi)^2}\Bigg[&- \frac{[16][23]^3[34]^3\langle \hat{2}\hat{3}\rangle\langle \hat{3}4\rangle(\langle \hat{1}5\rangle[15] + \langle 56\rangle[56])}{\langle 16\rangle\langle 45\rangle\langle 56\rangle^2\langle \hat{2}5\rangle\langle \hat{1}\hat{2}\rangle\langle \hat{1}4\rangle} \\
&+ \frac{[16][25]^3[45]^3\langle 45\rangle\langle \hat{2}5\rangle(\langle \hat{3}\hat{1}\rangle[31] + \langle \hat{3}6\rangle[36])}{\langle 16\rangle\langle \hat{2}\hat{3}\rangle\langle \hat{3}4\rangle\langle \hat{1}4\rangle\langle \hat{1}\hat{2}\rangle\langle \hat{3}6\rangle^2} \\
&+ \frac{[16][35][24]^3\langle \hat{2}4\rangle(\langle \hat{1}4\rangle[14] + \langle 46\rangle[46])^3}{\langle 16\rangle\langle 46\rangle^2\langle \hat{3}5\rangle\langle \hat{2}5\rangle\langle \hat{2}\hat{3}\rangle\langle \hat{1}5\rangle\langle \hat{3}\hat{1}\rangle} \\
&- \frac{[16][35][24]^3\langle \hat{2}4\rangle(\langle \hat{1}\hat{2}\rangle[12] + \langle \hat{2}6\rangle[26])^3}{\langle 16\rangle\langle 45\rangle\langle \hat{3}4\rangle\langle \hat{3}5\rangle\langle \hat{2}6\rangle^2\langle \hat{3}\hat{1}\rangle\langle \hat{1}5\rangle} \\
&+ \frac{[16][35](\langle \hat{1}4\rangle[14] + \langle 46\rangle[46])^3(\langle \hat{1}\hat{2}\rangle[12] + \langle \hat{2}6\rangle[26])^3}{\langle 16\rangle\langle 45\rangle\langle \hat{3}5\rangle\langle \hat{2}\hat{3}\rangle\langle \hat{2}5\rangle\langle \hat{3}4\rangle\langle \hat{2}6\rangle^2\langle \hat{1}4\rangle^2} \Bigg] \, .
\end{aligned}
\tag{2.12}
$$

We also include three other sets of terms which are the same as (2.12) but with $(2 \leftrightarrow 3)$,

$(4 \leftrightarrow 5)$, and with both $(2 \leftrightarrow 3)$ and $(4 \leftrightarrow 5)$,

$$\frac{8i}{(4\pi)^2}\left[\frac{[16][45][23]^3\langle\hat{2}\hat{3}\rangle(\langle\hat{3}\hat{1}\rangle[31]+\langle\hat{3}6\rangle[36])^3}{\langle16\rangle\langle45\rangle\langle\hat{2}4\rangle\langle\hat{2}5\rangle\langle\hat{1}4\rangle\langle\hat{1}5\rangle\langle\hat{3}6\rangle^2}\right.$$
$$\left.-\frac{[16][24]^3[25]^3\langle\hat{2}4\rangle\langle\hat{2}5\rangle(\langle\hat{3}\hat{1}\rangle[31]+\langle\hat{3}6\rangle[36])}{\langle16\rangle\langle\hat{3}4\rangle\langle\hat{3}5\rangle\langle\hat{1}4\rangle\langle\hat{1}5\rangle\langle\hat{3}6\rangle^2}\right]\;. \tag{2.13}$$

There is another set of terms which are the same as (2.13) but with $(2 \leftrightarrow 3)$,

$$\frac{8i}{(4\pi)^2}\left[\frac{[16][24]^3[34]^3\langle\hat{2}4\rangle\langle\hat{3}4\rangle(\langle\hat{1}5\rangle[15]+\langle56\rangle[56])}{\langle16\rangle\langle56\rangle^2\langle\hat{2}5\rangle\langle\hat{3}5\rangle\langle\hat{1}\hat{2}\rangle\langle\hat{3}\hat{1}\rangle}\right.$$
$$\left.+\frac{[16][23][45]^3\langle45\rangle(\langle\hat{1}5\rangle[15]+\langle56\rangle[56])^3}{\langle16\rangle\langle56\rangle^2\langle\hat{2}4\rangle\langle\hat{3}4\rangle\langle\hat{2}\hat{3}\rangle\langle\hat{1}\hat{2}\rangle\langle\hat{3}\hat{1}\rangle}\right]\;. \tag{2.14}$$

Furthermore, one has another set of terms which are the same as (2.13) but with $(4 \leftrightarrow 5)$,

$$\frac{8i}{(4\pi)^2}\left[\frac{[16][45](\langle\hat{1}\hat{2}\rangle[12]+\langle\hat{2}6\rangle[26])^3(\langle\hat{3}\hat{1}\rangle[31]+\langle\hat{3}6\rangle[36])^3}{\langle16\rangle\langle45\rangle\langle\hat{2}4\rangle\langle\hat{2}5\rangle\langle\hat{3}4\rangle\langle\hat{3}5\rangle\langle\hat{2}6\rangle^2\langle\hat{3}\hat{1}\rangle^2}\right.$$
$$\left.-\frac{[16][23](\langle\hat{1}4\rangle[14]+\langle46\rangle[46])^3(\langle\hat{1}5\rangle[15]+\langle56\rangle[56])^3}{\langle16\rangle\langle46\rangle^2\langle\hat{2}4\rangle\langle\hat{2}5\rangle\langle\hat{3}4\rangle\langle\hat{3}5\rangle\langle\hat{1}5\rangle^2\langle\hat{2}\hat{3}\rangle}\right]\;. \tag{2.15}$$

Summarising, (2.12) plus permutations contributes 20 terms, namely (2.13) plus permutations contributes 4 terms, (2.14) plus permutations contributes 4 terms and finally (2.15) contributes 2 terms. Thus we have a contribution from all 30 terms in the five-point all-plus amplitude.

The diagram in figure 4 is associated with $\langle\hat{1}6\rangle = 0$, or equivalently with $z = -\langle16\rangle/[23]\langle\eta6\rangle$. We have checked numerically that the sum of these terms agrees with the residue of the shifted known amplitude (2.1). Thus the known six point all-plus gravity amplitude is also correctly reproduced using recursive techniques. It seems likely that this approach will work for all of the all-plus amplitudes.

3. The one-loop $-+++$ gravity amplitude

We now turn to study the four-graviton one-loop amplitude with one negative helicity graviton. This case involves the new feature of double poles in the amplitude, which introduces complications into the recursion relations. It will be helpful to briefly review the known results for the gauge theory case before discussing gravity.

In [13] the five-, six- and seven-point one-loop Yang-Mills amplitudes with a single negative helicity gluon were derived from on-shell recursion relations. Unlike the all-plus amplitude of the previous section, these amplitudes contain a nonstandard factorisation onto a three-point all-plus vertex. The usual collinear limits in real Minkowski space allow us to derive the leading double-pole factorisation, but are not precise enough to also calculate the single pole underneath.
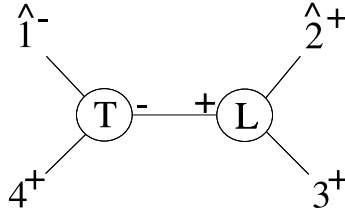
Figure 5: The diagram in the recursive expression for $A_4^{(1)}(1^-, 2^+, 3^+, 4^+)$.

Consider the simplest four-point case — the $-+++$ amplitude, first calculated in [30],

$$A_4^{(1)}(1^-, 2^+, 3^+, 4^+) = \frac{i}{96\pi^2} \frac{\langle 24 \rangle [24]^3}{[12]\langle 23 \rangle \langle 34 \rangle [41]} \ . \tag{3.1}$$

We will consider recursion based on the standard BCFW shifts on $|1]$ and $|2\rangle$:

$$\begin{aligned}
\lambda_1 &\to \lambda_1 \\
\widetilde{\lambda}_1 &\to \widetilde{\lambda}_1 - z\widetilde{\lambda}_2 \\
\lambda_2 &\to \lambda_2 + z\lambda_1 \\
\widetilde{\lambda}_2 &\to \widetilde{\lambda}_2 \ .
\end{aligned} \tag{3.2}$$

These shifts, when applied to the amplitude, do not give a boundary term and give a shifted amplitude which is singular at a single point in the complex z-plane,

$$\langle \hat{2}3 \rangle = \langle 13 \rangle (z - b) \quad , \quad [\hat{1}4] = [42](z - b) \, ,$$
$$\text{where} \quad b = -\frac{\langle 23 \rangle}{\langle 13 \rangle} = \frac{[14]}{[24]} \ . \tag{3.3}$$

Applying the shifts (3.2) to the known amplitude (3.1) yields

$$A_4^{(1)}(\hat{1}^-, \hat{2}^+, 3^+, 4^+)(z) = \frac{i}{96\pi^2} \frac{\langle 12 \rangle [24]}{\langle 34 \rangle \langle 31 \rangle} \left(\frac{1}{(z - b)^2} + \frac{\langle 13 \rangle \langle 14 \rangle}{\langle 34 \rangle \langle 12 \rangle} \frac{1}{(z - b)} \right) \ . \tag{3.4}$$

We can now write the original amplitude $A_4^{(1)}(\hat{1}^-, \hat{2}^+, 3^+, 4^+)(0)$ as a sum of residues of the poles that occur in the function $A_4^{(1)}(\hat{1}^-, \hat{2}^+, 3^+, 4^+)(z)/z$. In this case there is only one contribution, from the residue at $z = b$. Following [13], this single residue will be explained recursively by splitting it up into two parts. The first part comes from the double pole and the second from the single pole in (3.4). There will be a one-to-one map between the terms of this expansion and the terms of a recursion relation based on the shifts (3.2)

$$A_4^{(1)}(1^-, 2^+, 3^+, 4^+) = \frac{i}{96\pi^2} \left[\frac{\langle 12 \rangle \langle 13 \rangle [24]}{\langle 23 \rangle^2 \langle 34 \rangle} + \frac{\langle 12 \rangle \langle 13 \rangle [24]}{\langle 23 \rangle^2 \langle 34 \rangle} \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle} \right] \ . \tag{3.5}$$

We recall the origin of these two terms from a recursive diagram – both are associated with $\langle \hat{2}3 \rangle = [\hat{1}4] = 0$. The corresponding diagram in presented in figure 5.

BDK [13] reproduced the double-pole term in (3.5) recursively from the diagram in figure 5 using the one-loop all-plus vertex $V_3^{(1)}(+++)$ in (1.8)

$$A_3^{(0)}(4^+, \hat{1}^+, \hat{K}_{23}^-) \frac{i}{(K_{23}^2)^2} V_3^{(1)}(-\hat{K}_{23}^+, \hat{2}^+, 3^+) = \frac{i}{96\pi^2} \frac{\langle 1|\hat{K}|2]\langle 1|\hat{K}|3]}{\langle 14\rangle \langle 23\rangle^2 [23]} \frac{\langle \hat{K}1\rangle}{\langle \hat{K}4\rangle}$$

$$= \frac{i}{96\pi^2} \frac{\langle 12\rangle\langle 13\rangle[24]}{\langle 23\rangle^2 \langle 34\rangle} . \qquad (3.6)$$

The single pole under the double pole term in (3.5) differs from the double pole term by the factor

$$\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle} . \qquad (3.7)$$

The BDK correction factor introduced in (1.9) uses the soft functions given in equation (1.10) and in this case is equal to

$$S_3^{(0)}(\hat{1}, \hat{K}_{23}^+, 4)\, K_{23}^2\, S_3^{(0)}(\hat{2}, -\hat{K}_{23}^-, 3) = \frac{\langle 14\rangle\langle 23\rangle[23]^2}{\langle 1|\hat{K}|3]\langle 4|\hat{K}|2]} = \frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle} . \qquad (3.8)$$

We will now see how this approach applies to the gravity case.

The $-+++$ one-loop gravity amplitude which we will re-derive using a recursion relation is

$$M_4^{(1)}(1^-, 2^+, 3^+, 4^+) = \frac{i}{(4\pi)^2} \frac{1}{180} \left(\frac{\langle 12\rangle[23][24]}{[12]\langle 23\rangle\langle 24\rangle} \right)^2 (s^2 + st + t^2), \qquad (3.9)$$

and was calculated using string-based methods in [31] (we use the normalisation conventions of [28]).

Remarkably, the recursive procedure for Yang-Mills, reviewed in the last section, extends very simply to this gravity case. Just as in the Yang-Mills case, we consider the standard BCFW shifts on $|1]$ and $|2\rangle$ given in (3.2). Applying these shifts to the known amplitude does not give a boundary term and introduces singularities at two different points in the complex z-plane,

$$\langle \hat{2}4\rangle = \langle 14\rangle(z-a) \quad \text{where} \quad a = -\frac{\langle 24\rangle}{\langle 14\rangle} , \qquad (3.10)$$

$$\langle \hat{2}3\rangle = \langle 13\rangle(z-b) \quad \text{where} \quad b = -\frac{\langle 23\rangle}{\langle 13\rangle} . \qquad (3.11)$$

When we reconstruct this amplitude from a recursion relation the residues at these two points will come from different diagrams. Of course, there will be more recursive diagrams in gravity than there are in Yang-Mills, as in gravity there is no cyclic ordering of legs like there is for the colour ordered amplitudes of Yang-Mills.

Under the shifts (3.2), the amplitude (3.9) becomes

$$M_4^{(1)}(\hat{1}^-, \hat{2}^+, 3^+, 4^+)(z) = \frac{i}{(4\pi)^2} \frac{1}{180} \left[\frac{\langle 12\rangle^4 [23]^2 [24]^2}{\langle 13\rangle^2 \langle 14\rangle^2} \frac{1}{(z-a)^2(z-b)^2} \right.$$

$$+ \frac{\langle 12\rangle^3 [23]^3 [24]^2}{\langle 13\rangle\langle 14\rangle^2 [12]} \frac{1}{(z-a)^2(z-b)}$$

$$\left. + \frac{\langle 12\rangle^2 [23]^4 [24]^2}{\langle 14\rangle^2 [12]^2} \frac{1}{(z-a)^2} \right] . \qquad (3.12)$$
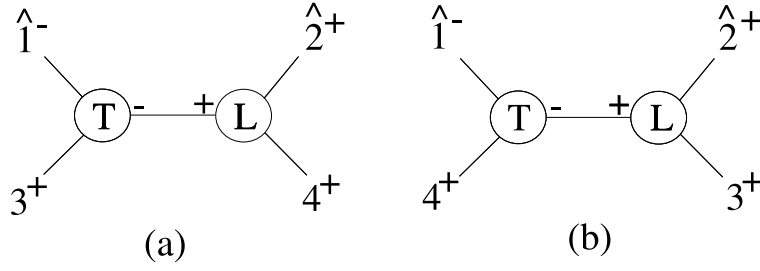
Figure 6: The two diagrams in the recursive expression for $M_4^{(1)}(1^-, 2^+, 3^+, 4^+)$.

We then separate out the different poles using partial fractions. The shifted amplitude is then expressed as a sum of terms associated with the various different types of pole at different locations,

$$M_4^{(1)}\big(\hat{1}^-, \hat{2}^+, 3^+, 4^+\big)(z) = \frac{i}{(4\pi)^2}\frac{1}{180}\left[\frac{\langle 12\rangle^2[23]^2[24]^2}{\langle 34\rangle^2}\frac{1}{(z-a)^2}\right.$$

$$+\frac{\langle 12\rangle\langle 13\rangle\langle 14\rangle[23]^2[24]^2}{\langle 43\rangle^3}\frac{1}{z-a} + \frac{\langle 12\rangle^2[23]^2[24]^2}{\langle 34\rangle^2}\frac{1}{(z-b)^2} + \left.\frac{\langle 12\rangle\langle 13\rangle\langle 14\rangle[23]^2[24]^2}{\langle 34\rangle^3}\frac{1}{z-b}\right] .$$

(3.13)

Finally, we can write the original amplitude $M_4^{(1)}\big(\hat{1}^-, \hat{2}^+, 3^+, 4^+\big)(0)$ as a sum of residues of the function $M_4^{(1)}\big(\hat{1}^-, \hat{2}^+, 3^+, 4^+\big)(z)/z$ at the poles of various types and locations in the complex z-plane:

$$M_4^{(1)} = \frac{i}{(4\pi)^2}\frac{1}{180}\left[\frac{\langle 12\rangle^2\langle 14\rangle^2[23]^2[24]^2}{\langle 24\rangle^2\langle 34\rangle^2}\right. \qquad \text{double-pole, } z=a \quad (3.14)$$

$$+\frac{\langle 12\rangle^2\langle 14\rangle^2[23]^2[24]^2}{\langle 24\rangle^2\langle 34\rangle^2}\left(-\frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 43\rangle}\right) \quad \text{single-pole, } z=a \quad (3.15)$$

$$+\frac{\langle 12\rangle^2\langle 13\rangle^2[23]^2[24]^2}{\langle 23\rangle^2\langle 34\rangle^2} \qquad \text{double-pole, } z=b \quad (3.16)$$

$$+\left.\frac{\langle 12\rangle^2\langle 13\rangle^2[23]^2[24]^2}{\langle 23\rangle^2\langle 34\rangle^2}\left(-\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}\right)\right] \quad \text{single-pole, } z=b \quad (3.17)$$

We will now reconstruct these four terms from the diagrams of a recursion relation. There will be two diagrams corresponding to the two locations, in the complex z-plane, where there are poles in the shifted amplitude $M_4^{(1)}\big(\hat{1}^-, \hat{2}^+, 3^+, 4^+\big)(z)$. The pole at $z=a$ is associated with $[\hat{1}3]=\langle\hat{2}4\rangle=0$ and corresponds to figure 6(a). The other pole, at $z=b$, is associated with $[\hat{1}4]=\langle\hat{2}3\rangle=0$ and corresponds to figure 6(b).

We now introduce a new three-point one-loop all-plus gravity vertex

$$W_3^{(1)}(1^+, 2^+, 3^+) = C([12][23][31])^2 , \qquad (3.18)$$

where C is a constant which we will fix shortly by comparison with the known answer (3.16). The double-pole term in (3.16) can then be reconstructed recursively from the diagram

corresponding to figure 6b using this. We have

$$M_3^{(0)}(\hat{1}^-, \hat{K}_{23}^-, 4^+)\frac{i}{(K_{23}^2)^2}W_3^{(1)}(\hat{2}^+, 3^+, -\hat{K}_{23}^+) = -C\frac{\langle 1|\hat{K}|3]^2\langle 1|\hat{K}|2]^2}{\langle 23\rangle^2\langle 41\rangle^2}\frac{\langle 1\hat{K}\rangle^2}{\langle 4\hat{K}\rangle^2} . \qquad (3.19)$$

We now use the following relations to write $\hat{K}$ in terms of the spinor variables of the external legs,

$$\langle 1|\hat{K}|3]^2 = \langle 1|\hat{2}+3|3]^2 = \langle 12\rangle^2[23]^2 ,$$
$$\langle 1|\hat{K}|2]^2 = \langle 1|\hat{2}+3|2]^2 = \langle 13\rangle^2[23]^2 ,$$
$$\frac{\langle 1\hat{K}\rangle^2}{\langle 4\hat{K}\rangle^2} = \frac{\langle 1|\hat{K}|2]^2}{\langle 4|\hat{K}|2]^2} = \frac{\langle 1|\hat{2}+3|2]^2}{\langle 4|\hat{2}+3|2]^2} = \frac{\langle 13\rangle^2}{\langle 34\rangle^2} . \qquad (3.20)$$

Thus (3.19) reproduces the spinor algebra of the known double pole residue at $z = b$ (3.16), giving

$$-C\frac{\langle 12\rangle^2\langle 13\rangle^2[23]^2[24]^2}{\langle 23\rangle^2\langle 34\rangle^2} . \qquad (3.21)$$

By comparison with (3.16) we can fix C in the new vertex $W_3^{(1)}(+++)$ to be:

$$C = -\frac{i}{(4\pi)^2}\frac{1}{180} . \qquad (3.22)$$

The other term (3.17), corresponding to figure 6b, is the residue of the single-pole underneath the double pole at $z = b$. This single-pole term differs from the double-pole term (3.16) and (3.19) up to a sign in the same way as in the Yang-Mills amplitude (3.5):

$$-\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle} = -S_3^{(0)}(\hat{1}, \hat{K}_{23}^+, 4)K_{23}^2 S_3^{(0)}(\hat{2}, -\hat{K}_{23}^-, 3) . \qquad (3.23)$$

Notice that the soft functions in (3.23) are those for Yang-Mills theory (explicitly written in (1.10)), rather than the gravity soft functions.

This remark leads us to suggest the following candidate for the single pole under the double pole in gravity:

$$-M_3^{(0)}(\hat{1}^-, \hat{K}_{23}^-, 4^+)S_3^{(0)}(\hat{1}, \hat{K}_{23}^+, 4)\frac{i}{K_{23}^2}S_3^{(0)}(\hat{2}, -\hat{K}_{23}^-, 3)W_3^{(1)}(\hat{2}^+, 3^+, -\hat{K}_{23}^+) . \qquad (3.24)$$

Figure 6a is the same as figure 6b, but with the external legs 3 and 4 interchanged. The two terms (3.14) and (3.15), associated with the residue at $z = a$, correspond to figure 6a, and are similarly found by interchanging legs 3 and 4.

4. The one-loop $-++++$ gravity amplitude

The $-++++$ one-loop gravity amplitude is unknown. We now discuss how one might construct it using on-shell recursion relations. First consider the shifts (3.2) on $|1]$ and $|2\rangle$; we assume the absence of a boundary term, as in Yang-Mills, where shifts of the form
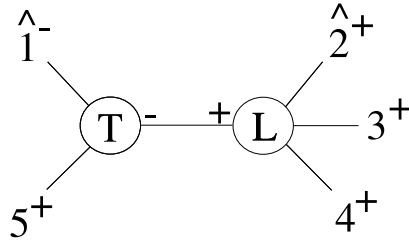
Figure 7: The diagram in the recursive expression for $M_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+)$ corresponding to a simple pole associated with $[\hat{1}5] = 0$.

$[-, +\rangle$ have been observed to be free of large-parameter contributions [16, 32]. Note that this observation extends to the $-+++$ gravity amplitude discussed above.

The shifts (3.2) give nine different recursive diagrams. The shifted amplitude has simple poles associated with $[\hat{1}3] = [\hat{1}4] = [\hat{1}5] = 0$. The simple pole associated with $[\hat{1}5] = 0$ corresponds to the standard recursive diagram in figure 7. The shifted amplitude will also have simple poles associated with $\langle\hat{2}3\rangle = \langle\hat{2}4\rangle = \langle\hat{2}5\rangle = 0$. The simple pole associated with $\langle\hat{2}3\rangle = 0$ corresponds to the standard factorisation diagram in figure 8. Finally there are also nonstandard factorisations in the shifted amplitude corresponding to the poles associated with $\langle\hat{2}3\rangle = \langle\hat{2}4\rangle = \langle\hat{2}5\rangle = 0$. These nonstandard factorisations contain a one-loop three-point all-plus vertex, and contribute double poles and also single poles under these double poles. The diagram for the nonstandard factorisation associated with the pole at $\langle\hat{2}3\rangle = 0$ is given in figure 9. There are just three types of diagram to calculate; figures 7, 8 and 9. The remaining diagrams can be obtained from these by permuting the external legs $\{3, 4, 5\}$.

To begin with, we consider the diagram in figure 7. Using similar manipulations to those detailed in previous sections, one finds that this contributes

$$-\frac{i}{(4\pi)^2}\frac{1}{60}\frac{\langle 15\rangle [25]^4}{\langle 34\rangle^2 [12]^2 [15]}\Big([23]^2[45]^2 + [23][34][45][25] + [34]^2[25]^2\Big). \qquad (4.1)$$

There is no colour ordering in gravity, so figure 7 is invariant under swapping the external legs labelled 3 and 4. Use of the Schouten identity shows that the result (4.1) is also invariant under swapping legs 3 and 4.

Next we consider figure 8. This contributes

$$\frac{i}{(4\pi)^2}\frac{1}{180}\frac{\langle 14\rangle^2\langle 15\rangle^2[23][45]^4}{\langle 12\rangle^2\langle 23\rangle\langle 35\rangle^2\langle 34\rangle^2\langle 45\rangle^2}\Big(\langle 14\rangle^2\langle 35\rangle^2 + \langle 14\rangle\langle 45\rangle\langle 53\rangle\langle 13\rangle + \langle 45\rangle^2\langle 13\rangle^2\Big). \quad (4.2)$$

The diagram in figure 8 is invariant under swapping the external legs labelled 4 and 5. Use of the Schouten identity shows that the result (4.2) is also invariant under swapping 4 and 5.

Finally we consider contributions from the diagram in figure 9. This diagram contains a three-point all-plus vertex, hence there will be two contributions here – a single and a double pole contribution, as we saw in the four-point example.
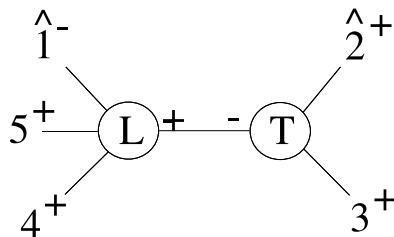
Figure 8: The diagram in the recursive expression for $M_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+)$ corresponding to a simple-pole associated with $\langle 23 \rangle = 0$.
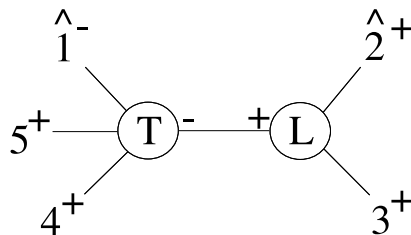

Figure 9: The diagram in the recursive expression for $M_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+)$ corresponding to the double-pole associated with $\langle \hat{2}3 \rangle = 0$.

First consider the double-pole term,

$$M_4^{(0)}(\hat{1}^-, \hat{K}_{23}^-, 4^+, 5^+) \frac{i}{(K_{23}^2)^2} W_3^{(1)}(\hat{2}^+, 3^+, -\hat{K}_{23}^+), \tag{4.3}$$

where the one-loop three-point all-plus vertex $W_3^{(1)}(+ + +)$ is the new vertex which was introduced in (3.18) and the $M_4^{(0)}(- - + +)$ amplitude is given via the following KLT relation

$$M_4^{(0)}(1^-, 2^-, 3^+, 4^+) = i\langle 12\rangle[12]A_4^{(0)}(1^-, 2^-, 3^+, 4^+)A_4^{(0)}(1^-, 2^-, 4^+, 3^+)$$

$$= -i\frac{\langle 12\rangle^7[12]}{\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2} . \tag{4.4}$$

Thus (4.3) yields

$$C\frac{\langle 1|\hat{K}|2]^2\langle 1|\hat{K}|3]^2}{\langle 14\rangle\langle 15\rangle\langle 23\rangle^2\langle 45\rangle^2}\langle 1|\hat{K}|\hat{1}]\frac{\langle 1\hat{K}\rangle\langle 1\hat{K}\rangle}{\langle 4\hat{K}\rangle\langle 5\hat{K}\rangle} . \tag{4.5}$$

Eliminating the hats in (4.5), we obtain

$$C\frac{\langle 12\rangle^2\langle 13\rangle^4[23]^4[45]}{\langle 14\rangle\langle 15\rangle\langle 23\rangle^2\langle 34\rangle\langle 35\rangle\langle 45\rangle} . \tag{4.6}$$

We recall that the coefficient C has been fixed in (3.22) by comparison with the known $-+++$ one-loop gravity amplitude. Finally, notice that figure 9 is invariant under swapping

the external legs labelled 4 and 5. The result (4.6) is also invariant under swapping 4 and 5.

The other contribution from figure 9 is from the "single-pole underneath the double-pole" term. Unfortunately this final term poses a problem. Inspired by the corresponding term (3.17) in the known $-+++$ gravity amplitude we might guess that this single-pole under the double-pole term differs from the double-pole term by a factor of the form introduced in (1.9), i.e.

$$S(a_1, \hat{K}^+, a_2) K^2 S(b_1, -\hat{K}^-, b_2) . \tag{4.7}$$

Experimentation in Yang-Mills [13] led to the conclusion that one should choose a_1 and a_2 to be the legs colour adjacent to the propagator in the tree-level amplitude of the recursive diagram. This prescription cannot extend directly to gravity however, since there is no colour ordering of the external legs. If we are to use a factor of this form in gravity we have to choose two of the three external legs attached to the tree diagram in figure (9) to be a_1 and a_2.

In the $-+++$ gravity amplitude we did not encounter this because the tree amplitude in the recursive diagram only has two external legs. However, since a factor of this form is antisymmetric under swapping a_1 and a_2, even in the $-+++$ gravity example, the lack of ordering of the external particles means that this factor has an ambiguous sign. For figure (9) there are three possible choices:

$$\begin{aligned} S(\hat{1}, \hat{K}_{23}^+, 4) K_{23}^2 S(\hat{2}, -\hat{K}_{23}^-, 3) &= -\frac{\langle 14 \rangle \langle 23 \rangle [23]^2}{\langle 1|\hat{K}|3] \langle 4|\hat{K}|2]} \\ &= \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle} , \end{aligned} \tag{4.8}$$

$$\begin{aligned} S(\hat{1}, \hat{K}_{23}^+, 5) K_{23}^2 S(\hat{2}, -\hat{K}_{23}^-, 3) &= -\frac{\langle 15 \rangle \langle 23 \rangle [23]^2}{\langle 1|\hat{K}|3] \langle 5|\hat{K}|2]} \\ &= \frac{\langle 15 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 35 \rangle} , \end{aligned} \tag{4.9}$$

$$\begin{aligned} S(5, \hat{K}_{23}^+, 4) K_{23}^2 S(\hat{2}, -\hat{K}_{23}^-, 3) &= \frac{\langle 23 \rangle \langle 45 \rangle [23]^2}{\langle 5|\hat{K}|2] \langle 4|\hat{K}|3]} \\ &= -\frac{\langle 13 \rangle \langle 23 \rangle \langle 45 \rangle}{\langle 12 \rangle \langle 34 \rangle \langle 35 \rangle} . \end{aligned} \tag{4.10}$$

It is perhaps natural to guess that a sum of these terms might give the correct single pole under the double pole term. Figure (9) is symmetric under swapping legs 4 and 5, so we require a sum of factors which share this symmetry. An appropriate sum of factors would be proportional to

$$\left(\frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle} + \frac{\langle 15 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 35 \rangle} \right) . \tag{4.11}$$

Collecting together all the previous expressions, and including this term, we are led to the

following proposal for this amplitude (a is a constant)

$$
M_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+)
$$
$$
= \frac{i}{(4\pi)^2}\frac{1}{180}\bigg(-3\frac{\langle 15\rangle[25]^4}{\langle 34\rangle^2[12]^2[15]}\Big([23]^2[45]^2 + [23][34][45][25] + [34]^2[25]^2\Big)
$$
$$
+ \frac{\langle 14\rangle^2\langle 15\rangle^2[23][45]^4}{\langle 12\rangle^2\langle 23\rangle\langle 35\rangle^2\langle 34\rangle^2\langle 45\rangle^2}\Big(\langle 34\rangle^2\langle 15\rangle^2 + \langle 13\rangle\langle 34\rangle\langle 45\rangle\langle 15\rangle + \langle 13\rangle^2\langle 45\rangle^2\Big)
$$
$$
+ \frac{\langle 12\rangle^2\langle 13\rangle^4[23]^4[45]}{\langle 14\rangle\langle 15\rangle\langle 23\rangle^2\langle 34\rangle\langle 35\rangle\langle 45\rangle}\Big(1 + a\frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle} + a\frac{\langle 15\rangle\langle 23\rangle}{\langle 12\rangle\langle 35\rangle}\Big)\bigg)
$$
$$
+ \frac{i}{(4\pi)^2}\frac{1}{180}\bigg(-3\frac{\langle 13\rangle[23]^4}{\langle 45\rangle^2[12]^2[13]}\Big([24]^2[53]^2 + [24][45][53][23] + [45]^2[23]^2\Big)
$$
$$
+ \frac{\langle 15\rangle^2\langle 13\rangle^2[24][53]^4}{\langle 12\rangle^2\langle 24\rangle\langle 43\rangle^2\langle 45\rangle^2\langle 53\rangle^2}\Big(\langle 45\rangle^2\langle 13\rangle^2 + \langle 14\rangle\langle 45\rangle\langle 53\rangle\langle 13\rangle + \langle 14\rangle^2\langle 53\rangle^2\Big)
$$
$$
+ \frac{\langle 12\rangle^2\langle 14\rangle^4[24]^4[53]}{\langle 15\rangle\langle 13\rangle\langle 24\rangle^2\langle 45\rangle\langle 43\rangle\langle 53\rangle}\Big(1 + a\frac{\langle 15\rangle\langle 24\rangle}{\langle 12\rangle\langle 45\rangle} + a\frac{\langle 13\rangle\langle 24\rangle}{\langle 12\rangle\langle 43\rangle}\Big)\bigg)
$$
$$
+ \frac{i}{(4\pi)^2}\frac{1}{180}\bigg(-3\frac{\langle 14\rangle[24]^4}{\langle 53\rangle^2[12]^2[14]}\Big([25]^2[34]^2 + [25][53][34][24] + [53]^2[24]^2\Big)
$$
$$
+ \frac{\langle 13\rangle^2\langle 14\rangle^2[25][34]^4}{\langle 12\rangle^2\langle 25\rangle\langle 54\rangle^2\langle 53\rangle^2\langle 34\rangle^2}\Big(\langle 53\rangle^2\langle 14\rangle^2 + \langle 15\rangle\langle 53\rangle\langle 34\rangle\langle 14\rangle + \langle 15\rangle^2\langle 34\rangle^2\Big)
$$
$$
+ \frac{\langle 12\rangle^2\langle 15\rangle^4[25]^4[34]}{\langle 13\rangle\langle 14\rangle\langle 25\rangle^2\langle 53\rangle\langle 54\rangle\langle 34\rangle}\Big(1 + a\frac{\langle 13\rangle\langle 25\rangle}{\langle 12\rangle\langle 53\rangle} + a\frac{\langle 14\rangle\langle 25\rangle}{\langle 12\rangle\langle 54\rangle}\Big)\bigg) \tag{4.12}
$$

However, one can check that this amplitude is not symmetric under the interchange of legs 2 and 3, for any values of the constant a, and hence cannot be the correct answer as it stands. We have also checked that it does not have all the correct collinear and soft limits for arbitrary a. Specifically, the (12), and (23), (24), (25) collinear limits are correct, as well as those (13), (14), (15) collinear limits involving the splitting functions $\mathrm{Split}_-^{\mathrm{gravity\ tree}}(i^+ j^-)$. However, other limits do not yield the correct results. Thus one concludes that the methods reviewed above fail to work in this case.

Recent papers may shed light on this problem, and suggest new approaches to solve it. The first is the general study on non-standard factorisations of [16] (any recursive diagram containing a three-point one-loop part is termed a "nonstandard" factorisation). It is these factorisations which are the complicating feature in the extension of the BCFW recursion relation from tree-level amplitudes to the rational parts of one-loop amplitudes. For example, factorisations involving the three-point all-plus amplitude give two types of term, a double pole and a single pole under the double pole term. While the description of the double pole in terms of a three-point all-plus vertex appears to be independent of the choice of shifts, the description of the single pole under the double pole in terms of a multiplicative correction factor SP^2S is not universal even in the Yang-Mills case, as we have checked in several cases. It only seems to work for the simplest BCFW shift on $|1]$ and $|2\rangle$.

We would like to illustrate this point by studying a specific example in Yang-Mills. We consider the $-+++$ Yang-Mills amplitude, which is given by

$$A_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+) = i\frac{N_p}{96\pi^2}\frac{1}{\langle 34\rangle^2}\left[-\frac{[25]^3}{[12][51]} + \frac{\langle 14\rangle^3[45]\langle 35\rangle}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} - \frac{\langle 13\rangle^3[32]\langle 42\rangle}{\langle 15\rangle\langle 54\rangle\langle 23\rangle^2}\right],$$
(4.13)

and we perform standard BCFW shifts on $|1]$ and $|3\rangle$. We then use partial fractions in order to separate the various poles. If we then set $z=0$, we have rewritten the amplitude in a form where there is a one-to-one correspondence between terms in this expansion and the terms of the recursion relation associated with the shifts:

$$A = i\frac{N_p}{96\pi^2}\left[\frac{[35]^3}{\langle 24\rangle^2[15][13]}\right.$$
(4.14)

$$+\frac{[23]^3}{\langle 45\rangle^2[12][13]}$$
(4.15)

$$+\frac{\langle 25\rangle\langle 14\rangle^3[45]}{\langle 13\rangle\langle 23\rangle\langle 45\rangle^2\langle 24\rangle^2}$$
(4.16)

$$-\frac{\langle 12\rangle^2\langle 13\rangle[23]}{\langle 45\rangle\langle 51\rangle\langle 24\rangle\langle 23\rangle^2}\left(1 + 2\frac{\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle}\right)$$
(4.17)

$$-\frac{\langle 25\rangle\langle 14\rangle^3[25]}{\langle 13\rangle\langle 34\rangle\langle 24\rangle^2\langle 45\rangle^2}$$
(4.18)

$$\left.-\frac{\langle 13\rangle\langle 14\rangle^3[43]}{\langle 12\rangle\langle 51\rangle\langle 24\rangle\langle 54\rangle\langle 34\rangle^2}\left(1 - 2\frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 42\rangle} + \frac{\langle 15\rangle\langle 34\rangle}{\langle 13\rangle\langle 45\rangle}\right)\right].$$
(4.19)

The diagrams in one-to-one correspondence with the above terms are given in figure 10. The terms (4.14)–(4.19) correspond to diagrams 10(a)– 10(f) respectively.

We find that the recursive description of these terms is well understood with the exception of the two factors relating the single pole under double pole terms to the corresponding double pole terms. Specifically, we require explanations of the factor $2\langle 14\rangle\langle 23\rangle/(\langle 13\rangle\langle 24\rangle)$ in (4.17) for the single pole under the double pole at $\langle 2\hat{3}\rangle=0$, and the factor $-2\langle 12\rangle\langle 34\rangle/(\langle 13\rangle\langle 42\rangle) + \langle 15\rangle\langle 34\rangle/(\langle 13\rangle\langle 45\rangle)$ in (4.19) for the single pole under the double pole at $\langle \hat{3}4\rangle=0$.

The spinor algebra in the single pole under double pole factor in (4.17) might be explained in a style similar to [13] by looking at the diagram in figure 10(d) and considering the legs colour adjacent to the propagator:

$$S(\hat{1}, k^+, 4)K_{23}S(2, k^-, \hat{3}) = \frac{\langle 14\rangle}{\langle 1\hat{k}\rangle\langle \hat{k}4\rangle}\langle 23\rangle[23]\frac{[23]}{[2\hat{k}][\hat{k}3]} = \frac{\langle 14\rangle\langle 23\rangle}{\langle 13\rangle\langle 24\rangle},$$
(4.20)

although the origin of the factor of 2 that appears in (4.17) is not clear.

The spinor algebra that appears in factors in (4.19) might similarly be explained, but for this diagram (figure 10(f)) we do not consider the colour adjacent legs to the propagator. The first factor in (4.19) is derived from

$$S(\hat{1}, k^+, 2)K_{34}S(\hat{3}, k^-, 4) = \frac{\langle 12\rangle}{\langle 1\hat{k}\rangle\langle \hat{k}2\rangle}\langle 34\rangle[34]\frac{[34]}{[3\hat{k}][\hat{k}4]} = \frac{\langle 12\rangle\langle 34\rangle}{\langle 13\rangle\langle 24\rangle},$$
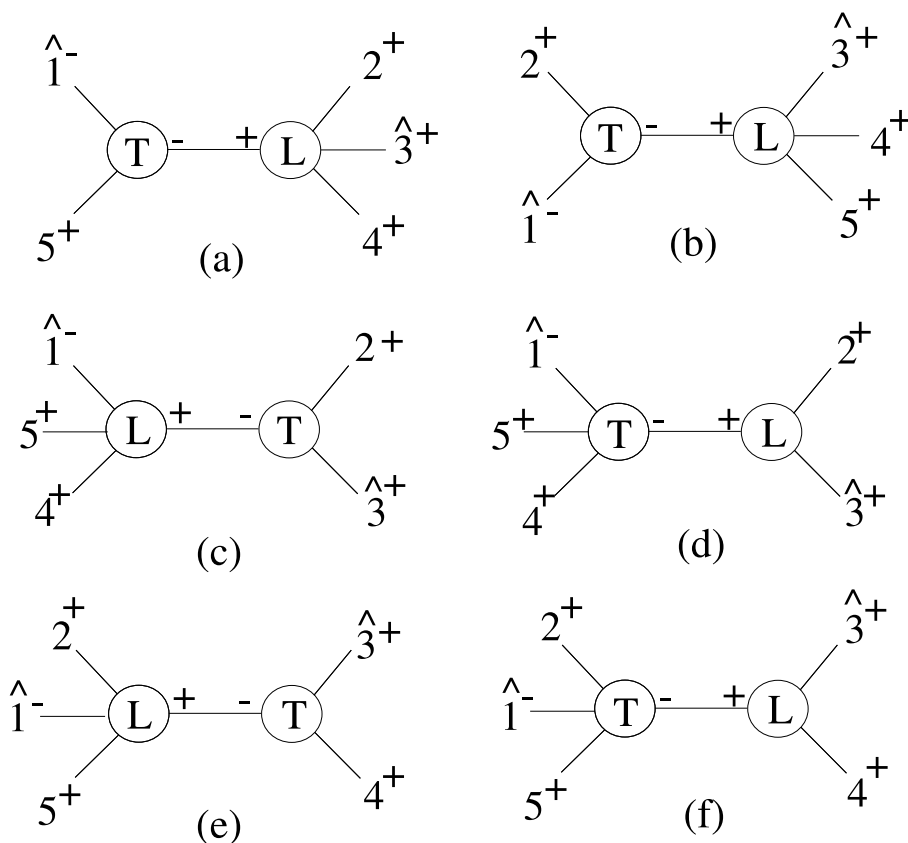(4.21)

Figure 10: The diagrams in the recursive expression for $A_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+)$.

and the second factor in (4.19) is derived from

$$S(\hat{1}, k^+, 5)K_{34}S(\hat{3}, k^-, 4) = \frac{\langle 15 \rangle}{\langle 1\hat{k} \rangle \langle \hat{k}2 \rangle} \langle 34 \rangle [34] \frac{[34]}{[3\hat{k}][\hat{k}4]} = \frac{\langle 15 \rangle \langle 34 \rangle}{\langle 13 \rangle \langle 45 \rangle} \; . \tag{4.22}$$

The other possibility is to completely avoid non-standard factorisations. In Yang-Mills it is possible to find shifts which do not involve non-standard factorisations (although these will generically lead to a boundary term) [16, 32]. Using a pair of shifts in two independent complex parameters, these authors exploited this fact to calculate complete amplitudes avoiding the consideration of any nonstandard factorisations.

We now briefly review their method for the simple case of a purely rational amplitude. The pair of shifts are called the primary shift and the auxiliary shift:

$$\text{primary shift: } [j, l\rangle \quad \begin{cases} \widetilde{\lambda}_j \to \widetilde{\lambda}_j - z\widetilde{\lambda}_l \\ \lambda_l \to \lambda_l + z\lambda_j \end{cases} \tag{4.23}$$

$$\text{auxiliary shift: } [a, b\rangle \quad \begin{cases} \widetilde{\lambda}_a \to \widetilde{\lambda}_a - w\widetilde{\lambda}_b \\ \lambda_b \to \lambda_b + w\lambda_a \end{cases} \tag{4.24}$$

The primary shift is chosen to have no non-standard factorisations, but it has a boundary term, while the auxiliary shift is chosen to have no boundary term, but it includes non-standard factorisations. These two shifts give rise to two recursion relations for the amplitude,

$$A_n^{(1)}(0) = \underset{[j,l\rangle}{\text{Inf}} \, A_n + R_n^{\text{D,recursive } [j,l\rangle} \,, \tag{4.25}$$

$$A_n^{(1)}(0) = R_n^{\text{D,recursive } [a,b\rangle} + R_n^{\text{D,non-standard } [a,b\rangle} \,. \tag{4.26}$$

We now apply the primary shift to the recursion relation for the auxiliary shift (4.26) to extract the large z behaviour of the primary shift:

$$\underset{[j,l\rangle}{\text{Inf}} \, A_n = \underset{[j,l\rangle}{\text{Inf}} \, R_n^{\text{D,recursive } [a,b\rangle} + \underset{[j,l\rangle}{\text{Inf}} \, R_n^{\text{D,non-standard } [a,b\rangle} \,, \tag{4.27}$$

where the Inf operation is defined to be the constant term in the expansion of the shifted term about $z = \infty$. We wish to avoid calculating terms involving nonstandard factorisations so we will assume that the following condition holds:

$$\underset{[j,l\rangle}{\text{Inf}} \, R_n^{\text{D,non-standard } [a,b\rangle} = 0 \,. \tag{4.28}$$

Since we do not, in general, know how to calculate the terms involving nonstandard factorisations, it is difficult to check explicitly if the condition (4.28) holds for a given pair of shifts. However, if one calculates an amplitude assuming that (4.28) holds and the resulting amplitude has the correct collinear and soft behaviour, then the amplitude is likely to be correct, and the property (4.28) must then have been true. Thus, if we assume the condition (4.28) and use (4.27) to calculate the boundary term in (4.25) we can calculate the amplitude without considering any nonstandard factorisations:

$$A_n(0) = \underset{[j,l\rangle}{\text{Inf}} \, R_n^{\text{D,recursive } [a,b\rangle} + R_n^{\text{D,recursive } [j,l\rangle} \,. \tag{4.29}$$

The following simple example exhibits the possibility of calculating an amplitude using auxiliary recursions to avoid all nonstandard factorisations (this is related to examples given in [16]). The three terms in the five point Yang-Mills amplitude (4.13) above will be called term 1, term 2 and term 3 for the purposes of this section. As shown in [13], if we consider the standard BCFW shifts on $|1]$ and $|2\rangle$ then term 1 and term 2 come from standard factorisations and term 3 comes from a nonstandard factorisation (see figure 11). Term 1 comes from the pole associated with $[\hat{1}5] = 0$, whilst terms 2 and 3 come from the pole associated with $\langle\hat{2}3\rangle = 0$. Term 2 is a standard factorisation, but term 3 involves the nonstandard three-point one-loop all-plus vertex. In [13] term 3 was computed by understanding this nonstandard factorisation as a sum of two terms called a double pole term and single pole under the double pole term.

Now we show how to use auxiliary recursions to calculate the amplitude without considering either of the two types of term associated with the three-point one-loop all-plus nonstandard factorisations. We will consider the following pair of shifts. The primary $[j,l\rangle$ shift is on $|4]$ and $|5\rangle$. This shift has no nonstandard factorisations, but does have boundary
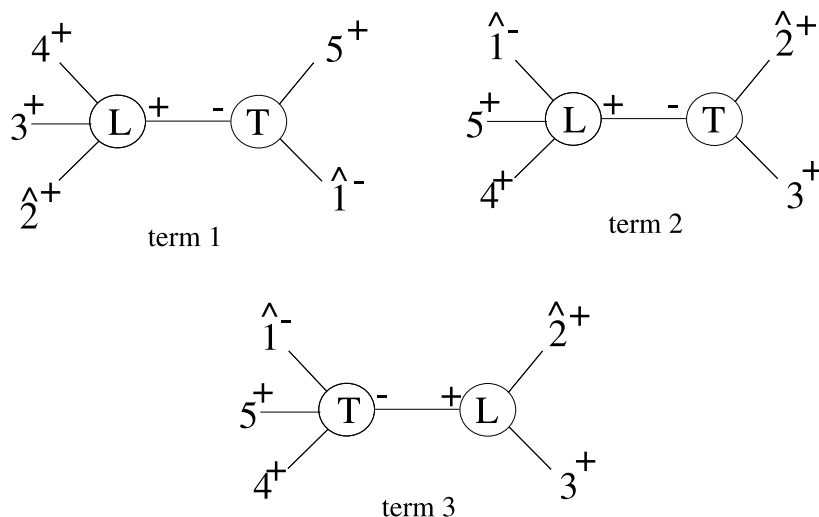
Figure 11: The diagrams in the $|1]\,|2\rangle$ shift of $A_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+)$.

term. The auxiliary $[a, b\rangle$ shift is on $|1]$ and $|2\rangle$. This shift has no boundary term, but does have nonstandard factorisations. From the discussion in the previous paragraph we known that

$$R_n^{\text{D,recursive } [a, b\rangle} = \text{term 1} + \text{term 2} \tag{4.30}$$

$$R_n^{\text{D,non-standard } [a, b\rangle} = \text{term 3} . \tag{4.31}$$

Since we are recalculating a known amplitude we can explicitly check if the condition (4.28) is satisfied. If we perform the $[j, l\rangle$ shift on term 3 (put hats on $|4]$ and $|5\rangle$) and then consider large z, then the term is $O(1/z)$ so the condition (4.28) is satisfied:

$$\operatorname*{Inf}_{[j,l\rangle} R_n^{\text{D,non-standard } [a, b\rangle} = \operatorname*{Inf}_{[j,l\rangle} \text{ term 3} = 0 . \tag{4.32}$$

Thus it will be possible to calculate the $-++++$ Yang-Mills amplitude without considering any nonstandard factorisations using this pair of shifts.

Now we summarise the details of actually calculating the amplitude. First we use (4.30) to calculate the first term in (4.29).

$$\operatorname*{Inf}_{[j,l\rangle} R_n^{\text{D,recursive } [a, b\rangle} = \operatorname*{Inf}_{[j,l\rangle} \left(\text{term 1} + \text{term 2} \right) = \text{term 1} + \text{term 2} . \tag{4.33}$$

As explained earlier, this term should be thought of as the boundary term in the primary shift. Finally we have to calculate the recursive diagrams in the primary $[j, l\rangle$ shift on $|4]$ and $|5\rangle$. There is only one diagram associated with these shifts. This is the diagram corresponding to a pole at $\langle 1\hat{5}\rangle = 0$ (see figure 12). Calculating this diagram gives the third term in (4.29)

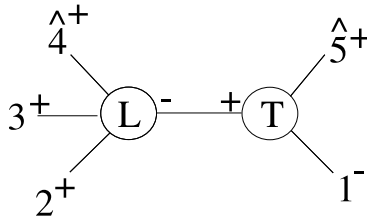$$R_n^{\text{D,recursive } [j, l\rangle} = \text{term 3} . \tag{4.34}$$

Figure 12: The diagram in the $|4] \, |5\rangle$ shift of $A_5^{(1)}(1^-, 2^+, 3^+, 4^+, 5^+)$.

Thus, putting (4.33) and (4.34) into the equation (4.29) constructs the full amplitude.

$$A_n(0) = \text{term } 1 + \text{term } 2 + \text{term } 3 \tag{4.35}$$

However, when we try to follow the above example in order to calculate the one-loop $-+++$ gravity amplitude this proves unsuccessful. In gravity, a shift generally involves more factorisations since there is no cyclic ordering condition on the external legs. Using the primary shift on $|4\rangle$ and $|5\rangle$ will not work for gravity since some of these extra factorisations are nonstandard. The shift on $|4\rangle$ and $|5\rangle$ involves the poles associated with $\langle \hat{5} 2 \rangle = 0$ and $\langle \hat{5} 3 \rangle = 0$, and these include contributions from the three-point one-loop all-plus factorisation.

In conclusion, we have seen that loop level recursion works for gravity in a number of cases, and provides relatively simple derivations of amplitudes. It seems likely that this will persist for the all-plus amplitudes in particular. However, in attempting to apply recursion to more complicated cases, such as the $-+++$ amplitude discussed in this section, one rapidly runs into difficulties. The currently known methods falter when confronted with the type of double-pole structures encountered here; one can derive straightforwardly some of the terms in the amplitude — such as those given in (4.12). In this case, these terms have a number of correct properties — one can check that some of the collinear and soft limits are correct for example. However, not all limits work, and neither do the required symmetries (we have also checked that the symmetrisation, in legs $(2, 3, 4, 5)$, of the expression (4.12) does not yield an expression with the right properties). Further contributions are missing, perhaps involving boundary terms. It is clear that what is needed is a complete understanding of non-standard factorisations in complex momenta and a general method of dealing with double poles.

Acknowledgments

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